On a pairing between symmetric power modules*

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Abstract

We prove, using purely combinatorial methods, that there is a pairing

$$\mathrm{Sym}^a\mathbb{Q}^2 \times \mathrm{Sym}^a\mathbb{Q}^2 \longrightarrow \mathbb{Q}$$

with an $M_2(\mathbb{Q})$-equivariance property.

Introduction

In associating Galois representations to modular eigenforms of weight k, one considers étale cohomology of modular curves with coefficients which are essentially ℓ-adic sheaves $\mathrm{Sym}^{k-2}\mathbb{Q}_\ell^2$. In order to prove properties of the Galois representations, we need to know as much as possible about these cohomology groups. It was asserted by Taylor ([2], p.270) that there is an explicit pairing $\langle\ ,\ \rangle : \mathrm{Sym}^a R^2 \times \mathrm{Sym}^a R^2 \longrightarrow R$ for any ring R with the property that $\langle x\alpha, y\alpha\rangle = \det(\alpha)^a \langle x, y\rangle$, where $\alpha \in M_2(R)$ has an induced right action on the symmetric power module. This was used there (and elsewhere) to give an explicit Poincaré duality on the étale cohomology groups, leading to a clearer understanding of the Galois representations.

However, as Kevin Buzzard pointed out to the first author, the pairing defined by Taylor does not actually satisfy the asserted property. In this note, we alter Taylor's definition slightly, and show, using entirely elementary combinatorial methods, that

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the desired property holds, at least if the ring has characteristic 0. Jordan and Livné ([1], Corollary 2.16) also seem to prove the existence of such a pairing, using properties of quaternion algebras, but the proof we give is rather more explicit, as well as being elementary.

1 The main theorem

For the moment, we will let R denote a $\mathbb{Q}$-algebra, so that every number is invertible. We let R^2 have standard basis $\{e_1, e_2\}$. Then there is an obvious action of a matrix $\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in M_2(R)$ on R^2 which we write on the right (as in [2]); the basis element $e_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$ is mapped to $\begin{pmatrix} \alpha_{11} & \alpha_{12} \end{pmatrix} = \alpha_{11}e_1 + \alpha_{12}e_2$, and similarly e_2 is mapped to $\alpha_{21}e_1 + \alpha_{22}e_2$. This action induces an action on $\mathrm{Sym}^a R^2$, where we take the standard basis $\{e_1^{\otimes a}, e_1^{\otimes a-1} \otimes e_2, \ldots, e_2^{\otimes a}\}$; the basis element $e_1^{\otimes i} \otimes e_2^{\otimes a-i}$ is mapped to $(\alpha_{11}e_1 + \alpha_{12}e_2)^{\otimes i} \otimes (\alpha_{21}e_1 + \alpha_{22}e_2)^{\otimes a-i}$, and we extend this linearly to all elements of $\mathrm{Sym}^a R^2$. (One can think of $\mathrm{Sym}^a R^2$ as the homogeneous polynomials of degree a, spanned by $x^a, x^{a-1}y, \ldots, y^a$.)

Define the matrix W to be the $(a+1) \times (a+1)$-matrix, where we index the rows and columns from $0, \ldots, a$, by:

$$W_{ij} = \begin{cases} 0, & \text{if } i + j \neq a \\ (-1)^i \binom{a}{i}^{-1}, & \text{if } i + j = a. \end{cases}$$

Define the pairing

$$\begin{array}{rcl} \mathrm{Sym}^a R^2 \times \mathrm{Sym}^a R^2 & \longrightarrow & R \\ (x, y) & \mapsto & xWy^t, \end{array}$$

where, as above, we think of elements as row vectors with respect to the standard basis. With this notation, we claim that this pairing satisfies the required property:

Theorem 1.1 *Let $x, y \in \mathrm{Sym}^a R^2$. Then for any matrix $\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in M_2(R)$,*

$$\langle x\alpha, y\alpha \rangle = \det(\alpha)^a \langle x, y \rangle.$$

Proof. Let us write A for the $(a+1) \times (a+1)$-matrix giving the action of α on $\mathrm{Sym}^a R^2$ with respect to the standard basis.

A short, elementary calculation shows that A_{kl} is given by

$$\sum_{t=0}^{a} \binom{a-k}{t} \binom{k}{l-t} \alpha_{11}^{a-k-t} \alpha_{12}^{t} \alpha_{21}^{k+t-l} \alpha_{22}^{l-t},$$

where we again index the rows and columns from 0 to a. Then the claim is equivalent to

$$AWA^t = d^a W,$$

where $d = \det\,\alpha = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$.

It is easy to see, however, that this equality is equivalent to the equality $A^tW^{-1}A = d^aW^{-1}$. It therefore suffices to prove something similar for the transpose matrix. The entries of W^{-1} are just like those of W, except that the klth entry is $(-1)^{a-k}\binom{a}{k}$ if $k+l = a$.

We need to see that if $i + j \neq a$, then the ijth entry of $A^tW^{-1}A$ vanishes, and if $i + j = a$, then the ijth entry is $d^a(-1)^{a-i}\binom{a}{i}$.

Using the expression for A_{kl}, a calculation gives the ijth entry of $A^tW^{-1}A$ to be

$$\sum_{r=0}^{a} \alpha_{11}^{a-r}\alpha_{12}^{r}\alpha_{21}^{a+r-i-j}\alpha_{22}^{i+j-r}\left[\sum_{m=0}^{a}(-1)^{a-m}\binom{a}{m}\sum_{s=0}^{a}\binom{a-m}{s}\binom{m}{i-s}\binom{m}{r-s}\binom{a-m}{j+s-r}\right].$$

Write the term $\alpha_{11}^{a-r}\alpha_{12}^{r}\alpha_{21}^{a+r-i-j}\alpha_{22}^{i+j-r}$ as $\alpha_{11}^{a}\alpha_{21}^{a-i-j}\alpha_{22}^{i+j}z^r$, where z is $\frac{\alpha_{12}\alpha_{21}}{\alpha_{11}\alpha_{22}}$.

Then our sum can be rewritten as

$$\alpha_{11}^{a}\alpha_{21}^{a-i-j}\alpha_{22}^{i+j}\sum_{m=0}^{a}(-1)^{a-m}\binom{a}{m}\sum_{r=0}^{a}\sum_{s=0}^{a}z^r\binom{a-m}{s}\binom{m}{i-s}\binom{m}{r-s}\binom{a-m}{j+s-r},$$

or, writing $t = r - s$,

$$\alpha_{11}^{a}\alpha_{21}^{a-i-j}\alpha_{22}^{i+j}\sum_{m=0}^{a}(-1)^{a-m}\binom{a}{m}\sum_{t=0}^{a}\sum_{s=0}^{a}z^{s+t}\binom{a-m}{s}\binom{m}{i-s}\binom{m}{t}\binom{a-m}{j-t}.$$

It is easy to see that the inner double sum is just the coefficient of x^iy^j in the product $(1+xz)^{a-m}(1+x)^m(1+yz)^m(1+y)^{a-m}$. Therefore the sum

$$\sum_{m=0}^{a}(-1)^{a-m}\binom{a}{m}\sum_{t=0}^{a}\sum_{s=0}^{a}z^{s+t}\binom{a-m}{s}\binom{m}{i-s}\binom{m}{t}\binom{a-m}{j-t}$$

is the coefficient of x^iy^j in

$$\begin{aligned}&\sum_{m=0}^{a}(-1)^{a-m}\binom{a}{m}(1+xz)^{a-m}(1+x)^m(1+yz)^m(1+y)^{a-m}\\ =\;&[(1+x)(1+yz)-(1+xz)(1+y)]^a\end{aligned}$$

by the binomial formula. But

$$(1+x)(1+yz)-(1+xz)(1+y) = (1-z)(x-y),$$

so the coefficient of x^iy^j in $[(1+x)(1+yz)-(1+xz)(1+y)]^a$ is the same as the coefficient of x^iy^j in $(1-z)^a(x-y)^a$. But clearly there are no terms except in degree a, so unless $i + j = a$, the sum vanishes, as required.

If $i + j = a$, the sum

$$\alpha_{11}^{a}\alpha_{21}^{a-i-j}\alpha_{22}^{i+j}\sum_{m=0}^{a}(-1)^{a-m}\binom{a}{m}\sum_{t=0}^{a}\sum_{s=0}^{a}z^{s+t}\binom{a-m}{s}\binom{m}{i-s}\binom{m}{t}\binom{a-m}{j-t}$$

becomes the coefficient of $x^i y^j$ in

$$\alpha_{11}^a \alpha_{22}^a (1-z)^a (x-y)^a.$$

But

$$1 - z = 1 - \frac{\alpha_{12}\alpha_{21}}{\alpha_{11}\alpha_{22}} = \frac{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}}{\alpha_{11}\alpha_{22}} = \frac{d}{\alpha_{11}\alpha_{12}},$$

so that $\alpha_{11}^a \alpha_{22}^a (1-z)^a$ is just d^a, and we need the coefficient of $x^i y^j$ in $d^a (x-y)^a$, namely $d^a (-1)^{a-i} \binom{a}{i}$, which is d^a multiplied by the ijth entry of W^{-1}, exactly as required. □

Remark 1.2 We stated this as a result for $\mathbb{Q}$-algebras, for simplicity. We only require that the binomial coefficients $\binom{a}{0}, \binom{a}{1}, \ldots, \binom{a}{a}$ should be invertible in R. Since no prime greater than a divides any of these binomial coefficients, we see that the result holds for any ring R in which primes up to a are invertible. In particular, the claim holds if R is any field of characteristic greater than a. (We should remark that Taylor [2] uses this claim only for rings which have this property, and the results of his paper therefore remain valid.)

Now that we know that the identity for $A^t W^{-1} A$ holds, the equivalent result for AWA^t implies the following identity for binomial coefficients: If the numbers a, i, j and r are given, then

$$\sum_{m=0}^{a} \frac{(-1)^m}{\binom{a}{m}} \left[\sum_{s=0}^{a} \binom{a-i}{s} \binom{i}{m-s} \binom{a-j}{a+r-i-j-s} \binom{j}{i+j+s-r-m} \right],$$

or, equivalently, its obvious rearrangement

$$\sum_{s=0}^{a} \binom{a-i}{s} \binom{a-j}{a+r-i-j-s} \left[\sum_{m=0}^{a} \frac{(-1)^m}{\binom{a}{m}} \binom{i}{m-s} \binom{j}{i+j+s-r-m} \right],$$

is equal to 0 if $i + j \neq a$, and if $i + j = a$, it is

$$(-1)^{i+r} \frac{\binom{a}{r}}{\binom{a}{i}} = (-1)^{i+r} \frac{i!j!}{r!(a-r)!}.$$

This follows because this is a coefficient of one of the monomials in the ijth entry of AWA^t.

References

[1] B. Jordan, R. Livné, Integral Hodge theory and congruences betwen modular forms, Duke Math. J. 80 (1995) 419–484

[2] R. Taylor, On Galois representations associated to Hilbert modular forms, Invent. math. 98 (1989) 265–280