A DISTRIBUTION RELATION ON ELLIPTIC CURVES*

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Abstract

In this paper, we give an elementary proof of a curious identity of elliptic functions. It is very similar to a beautiful proof given by Coates of a different identity. The result was strongly motivated by Wildeshaus's generalisation of Zagier's conjecture.

Introduction

Suppose that G is an abelian group. In our applications, we will have in mind the case $G = \mathbb{C}$, the complex numbers under addition. We will write the group law on G additively, and denote the identity by 0. Write

$$\mathrm{Pic}^0(G) = \{\sum_{g\in G} n_g(g) | \sum_{g\in G} n_g = 0 \text{ and } n_g = 0 \text{ for all but finitely many } g\}.$$

Thus $\mathrm{Pic}^0(G)$ denotes the degree 0 part of the integral group ring (the augmentation ideal). There is a natural homomorphism

$$\begin{aligned} q_r : \mathrm{Pic}^0(G)\otimes\mathbb{Q} &\longrightarrow \mathrm{Sym}^r(G)\otimes\mathbb{Q} \\ (x)-(0) &\mapsto x\otimes\cdots\otimes x\otimes 1 \end{aligned}$$

*AMS subject classification: 11G16, 33E05; secondary: 19F15, 19F27 and extend it linearly to all of the divisors. One has the following result:

PROPOSITION 0.1 ([2], LEMMA 2.4; [6], LEMME 4.1) *The kernel of* q_2 *is generated by symbols of the form*

$$(x, y) = (x + y) + (x - y) - 2(x) - 2(y) + 2(0)$$

for x, $y \in G$.

Thus, in some sense, the relation

$$(x + y)^2 + (x - y)^2 = 2x^2 + 2y^2$$

is the "only" linear relationship between squares of linear forms. The corresponding question for $r > 2$ seems unresolved.

We fix a $\mathbb{Z}$-lattice L, and concentrate attention on the case $G = \mathbb{C}$ as mentioned above. We recall the well-known Siegel function,

$$\varphi_L(z) = \varphi(z, L) = e^{-z\eta(z,L)/2}\sigma(z, L)\Delta(L)^{1/12},$$

defined for $z \in \mathbb{C}\backslash L$, and extend its definition linearly to divisors $D = \sum_{i=1}^{r} a_i(z_i)$ in $\mathbb{C}$ by

$$\varphi_L(D) = \prod_{\substack{i=1 \\ z_i \notin L}}^{r} \varphi_L(z_i)^{a_i}.$$

Unfortunately, the Siegel function is not an elliptic function, and so this formula does not naturally descend to the elliptic curve $\mathbb{C}/L$. However, given a divisor D on $\mathbb{C}/L$, we define $\varphi_L(D)$ to be $\varphi_L(\widetilde{D})$, where $\widetilde{D} = \sum_{i=1}^{r} a_i(z_i)$

is a divisor on $\mathbb{C}$ such that $\sum_{i=1}^r a_i z_i \otimes z_i = 0$ in $\mathbb{C} \otimes \mathbb{C}$. I am grateful to Norbert Schappacher for pointing out the remark (due to Jörg Wildeshaus) that this is the correct choice of lift of D to $\mathbb{C}$ in this situation.

Then we prove the following:

THEOREM 0.2 *For any isogeny of elliptic curves $\mathbb{C}/L \xrightarrow{\psi} \mathbb{C}/L'$ and any divisor D in $\ker(q_2)$, we have up to a root of unity,*

$$\varphi_{L'}(D) = \prod_{w \in \ker \psi} \varphi_L(D \oplus w)$$

where $D \oplus w$ denotes the translate of the divisor D obtained by adding w to every point in its support.

Throughout the paper, the symbol $=$ will indicate equality of two complex numbers only up to a root of unity; we are therefore really working in the ring $\mathbb{C}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$. Write also $\mathcal{O}$ for the identity in $\mathbb{C}/L$.

For the proof of this result, we will have to deal separately with several cases. As $\ker(q_2)$ is generated by symbols of the form (x, y), it suffices to consider divisors of this form. Generically, x and y do not lie in L, but the form of $\varphi_L(D)$ is slightly different when this fails to hold (as $\varphi_L(D)$ only deals with the points in the support of D not in L), and so we treat the other cases separately. In particular, corresponding to the cases

$$x \neq \pm y \text{ in } \mathbb{C}/L$$

$$x = y, x \neq -y \text{ in } \mathbb{C}/L$$

$$x \neq y, x = -y \text{ in } \mathbb{C}/L$$

$$x = y, x = -y \text{ in } \mathbb{C}/L,$$

we have respectively the symbols

$$\begin{aligned}
\{x, y\} &= (x+y) + (x-y) - 2(x) - 2(y) \\
\{x, x\} &= (2x) - 4(x) \text{ if } 2x \neq \mathcal{O} \text{ in } \mathbb{C}/L \\
\{x, -x\} &= (2x) - 2(x) - 2(-x) \text{ if } 2x \neq \mathcal{O} \text{ in } \mathbb{C}/L \\
\{x, x\} &= -4(x) \text{ if } 2x = \mathcal{O} \text{ in } \mathbb{C}/L
\end{aligned}$$

The first section of this paper provides a proof of the theorem for the symbol $\{x, y\}$, and the other section proves the result for the other symbols.

This work would not have been possible without many discussions with Klaus Rolshausen and Norbert Schappacher; I should like to thank them very much for these, and also for their hospitality during my visits to Strasbourg. Indeed, that the theorem should hold was suggested by Rolshausen [6], who uses a version of the above result, together with all of his results, in his work on the lowest step of Wildeshaus's generalisation of Zagier's conjecture to the case of elliptic curves (the proof given in this paper was first outlined in a letter of January 17th, 1996 from the author to Rolshausen). We would not be able to improve on his presentation, nor that of Wildeshaus himself ([8] and [9]), and so we make no attempt to introduce the formalism of Wildeshaus. We note, however, that Wildeshaus has also proven a version of the theorem above for elliptic curves over an arbitrary base scheme, using different and very much more sophisticated techniques, in [9]. For similar distribution relations, we also refer the reader to [3]. We offer these proofs in the hope that they are of interest in their own right, and that the techniques used here may prove more intricate distribution relations.

1 The generic situation

The proof generalises a proof due to Coates and Taylor appearing in the appendix to [1].

Write

$$\zeta(z,w,t;L) = \frac{\varphi(z+w+t,L)\varphi(z-w+t,L)}{\varphi(z+t,L)^2\varphi(w+t,L)^2}.$$

In particular

$$\zeta(z,w,0;L) = (\wp_L(w) - \wp_L(z))\Delta(L)^{-1/6}.$$

First let $\beta \in \mathrm{End}(\mathbb{C}/L)$. Put

$$\omega(\beta) = \frac{\zeta(\beta z, \beta w, 0; L)}{\prod_{t\in\ker\beta}\zeta(z,w,t;L)}.$$

This is well-defined in $\mathbb{C}^\times \otimes \mathbb{Q}$. $\omega(\beta)$ is meromorphic of divisor 0 so that $\omega(\beta)$ is constant.

One shows that there exists a constant $c(L)$ such that

$$\omega(\beta) = \frac{\zeta(\beta z, \beta w, 0; L)}{\prod_{t \in \ker \beta} \zeta(z, w, t; L)} = c(L)^{\deg \beta - 1}.$$

If such a constant exists, it is unique: if $c(L)$ and $c'(L)$ are two choices, one considers the two endomorphisms $\beta = [2]$, $\beta = [3]$ to find

$$\begin{aligned} c(L)^3 &= c'(L)^3 \\ c(L)^8 &= c'(L)^8 \end{aligned}$$

so that $c(L) = c'(L)$.

Lemma 1.1 *If β_1, $\beta_2 \in \mathrm{End}(\mathbb{C}/L)$, one has*

$$\omega(\beta_2\beta_1) = \omega(\beta_1)^{\deg \beta_2} \omega(\beta_2).$$

Proof. Consider the exact sequence

$$0 \longrightarrow \ker \beta_1 \longrightarrow \ker(\beta_2\beta_1) \xrightarrow{\beta_1} \ker \beta_2 \longrightarrow 0.$$

Let H be a set of representatives of $\ker \beta_2$ in $\ker(\beta_2\beta_1)$. Then each point $t \in \ker(\beta_2\beta_1)$ may be written as $r + s$, with $r \in H$, $s \in \ker \beta_1$. Then:

$$\sum_{r_1, r_2 \in H} \sum_{s_1, s_2 \in \ker \beta_1} \{x + r_1 + s_1, y + r_2 + s_2\}$$

$$
\begin{aligned}
&= \sum_{r_1,r_2\in H}\left[\deg\beta_1 \sum_{s\in\ker\beta_1}\{\{x+r_1,y+r_2\}\oplus s\}\right]\\
&= \deg(\beta_2\beta_1)\sum_{r\in H}\sum_{s\in\ker\beta_1}\{\{x,y\}\oplus(r+s)\}
\end{aligned}
$$

One calculates

$$\prod_{r_1,r_2\in H}\prod_{s_1,s_2\in\ker\beta_1}\zeta(x+r_1+s_1,y+r_2+s_2,0;L)$$

in two different ways. First, one has

$$\prod_{r_1,r_2\in H}\prod_{s_1,s_2\in\ker\beta_1}\zeta(x+r_1+s_1,y+r_2+s_2,0;L)$$

$$
\begin{aligned}
&= \left[\prod_{r\in H}\prod_{s\in\ker\beta_1}\zeta(x,y,r+s;L)\right]^{\deg\beta_2\beta_1}\\
&= [\omega(\beta_2\beta_1).\zeta(\beta_2\beta_1 x,\beta_2\beta_1 y,0;L)]^{\deg\beta_2\beta_1}
\end{aligned}
$$

On the other hand,

$$\prod_{r_1,r_2\in H}\prod_{s_1,s_2\in\ker\beta_1}\zeta(x+r_1+s_1,y+r_2+s_2,0;L)$$

$$
\begin{aligned}
&= \prod_{r_1,r_2\in H}\left[\prod_{s\in\ker\beta_1}\zeta(x+r_1,y+r_2,s;L)\right]^{\deg\beta_1}\\
&= \prod_{r_1,r_2\in H}[\omega(\beta_1).\zeta(\beta_1(x+r_1),\beta_1(y+r_2),0;L)]^{\deg\beta_1}\\
&= \omega(\beta_1)^{\deg(\beta_1)\deg(\beta_2)^2}\left[\prod_{r\in\ker\beta_2}\zeta(\beta_1 x,\beta_1 y,r;L)\right]^{\deg(\beta_1)\deg(\beta_2)}\\
&= \omega(\beta_1)^{\deg(\beta_1)\deg(\beta_2)^2}[\omega(\beta_2).\zeta(\beta_2\beta_1 x,\beta_2\beta_1 y,0;L)]^{\deg(\beta_1)\deg(\beta_2)}
\end{aligned}
$$

One then deduces that

$$\omega(\beta_2\beta_1)^{\deg(\beta_1)\deg(\beta_2)} = \omega(\beta_1)^{\deg(\beta_1)\deg(\beta_2)^2}\omega(\beta_2)^{\deg(\beta_1)\deg(\beta_2)}$$

and the result is immediate.

Corollary 1.2 *One has*

$$\omega(\beta_1)^{\deg(\beta_2)-1} = \omega(\beta_2)^{\deg(\beta_1)-1}$$

Proof. One has $\beta_1\beta_2 = \beta_2\beta_1$, as the ring $\mathrm{End}(E)$ is commutative for elliptic curves E. Then $\omega(\beta_1\beta_2) = \omega(\beta_2\beta_1)$; use the lemma to complete the proof.

Corollary 1.3 $\omega(\beta) = c(L)^{\deg(\beta)-1}$.

Proof. Choose μ_1, $\mu_2 \in \mathrm{End}(\mathbb{C}/L)$ with $(\deg(\mu_1)-1, \deg(\mu_2)-1) = 1$. For example, one can choose $\mu_1 = [2]$, $\mu_2 = [3]$. Then there exist n_1, n_2 with

$$n_1(\deg(\mu_1)-1) + n_2(\deg(\mu_2)-1) = 1.$$

One applies the last corollary, with β and μ_1 and also with β and μ_2. One finds

$$\omega(\beta) = (\omega(\mu_1)^{n_1}\omega(\mu_2)^{n_2})^{\deg(\beta)-1}.$$

If one writes

$$\gamma(z,w,t;L) = c(L)\zeta(z,w,t;L),$$

the lemma gives:

COROLLARY 1.4 $\gamma(\beta z, \beta w, 0; L) = \prod_{t \in \ker \beta} \gamma(z, w, t; L)$.

THEOREM 1.5 *$c(L)^{12} = 1$, so that $c(L)$ is trivial in $\mathbb{C}^\times \otimes \mathbb{Q}$.*

Proof. We compute $c(L)$ by making a particularly convenient choice of the endomorphism β. As in [1], we choose $\beta = [2]$. Then we know that

$$\omega([2]) = \frac{\zeta(2z, 2w, 0; L)}{\prod_{t \in \ker[2]} \zeta(z, w, t; L)} = c(L)^3.$$

One sees easily (by rearranging the terms of the product) that

$$\prod_{t \in \ker[2]} \zeta(z, w, t; L)^4 = \prod_{s_1, s_2 \in \ker[2]} \zeta(z + s_2, w + s_1, 0; L).$$

Now we recall that

$$\zeta(z, w, 0; L) = (\wp(w) - \wp(z))\Delta(L)^{-1/6}$$

(where $\wp(z)$ is defined with reference to L). Let N (resp. D) denote the numerator and denominator of $\omega([2])^4 = c(L)^{12}$. Thus

$$N = (\wp(2w) - \wp(2z))^4 \Delta(L)^{-2/3}$$

and

$$D = \prod_{s_1, s_2 \in \ker[2]} (\wp(w + s_1) - \wp(z + s_2)).\Delta(L)^{-8/3}.$$

The quotient $\frac{N}{D}$ is $c(L)^{12}$, so constant. To evaluate it, we consider the asymptotic behaviour of N and D as w and z approach 0. So suppose w and z are very small. Then

$$\begin{aligned} N &\sim \left(\frac{1}{(2w)^2} - \frac{1}{(2z)^2}\right)^4 \Delta(L)^{-2/3} \\ &= \left(\frac{z^2 - w^2}{w^2 z^2}\right)^4 4^{-4}\Delta(L)^{-2/3} \end{aligned}$$

Now we compute the asymptotics of D. Let $\{0, t_1, t_2, t_3\}$ be representatives for ker[2]. Then

$$\begin{aligned} D &\sim \left(\frac{1}{w^2} - \frac{1}{z^2}\right)\left(\frac{1}{w^2}\right)^3\left(-\frac{1}{z^2}\right)^3 .\Delta(L)^{-8/3} \\ &\qquad \prod_{s_1, s_2 \in \ker[2]-\{0\}} (\wp(w+s_1) - \wp(z+s_2)). \end{aligned}$$

Suppose

$$\wp(w + t_i) = \wp(t_i) + c_i w^2 + \text{higher order terms}$$

for small w. Write also

$$C_i = \prod_{j \neq i} (\wp(t_i) - \wp(t_j)).$$

Then, for $k = 1, 2, 3$,

$$\begin{aligned} &(\wp(w+t_k) - \wp(z+t_1))(\wp(w+t_k) - \wp(z+t_2))(\wp(w+t_k) - \wp(z+t_3)) \\ &\sim \prod_{i=1}^{3} (\wp(t_k) - \wp(t_i) + c_k w^2 - c_i z^2) \\ &= \prod_{i=1}^{3} (\wp(t_k) - \wp(t_i)) + \sum_{i=1}^{3} (c_k w^2 - c_i z^2) \prod_{j \neq i} (\wp(t_k) - \wp(t_j)) + \text{h.o.t.} \\ &= c_k(w^2 - z^2) C_k + \text{h.o.t.}, \end{aligned}$$

terms vanishing as $t_k = t_i$ for some i. It follows that

$$\begin{aligned} D &\sim \left(\frac{1}{w^2} - \frac{1}{z^2}\right)\left(\frac{1}{w^2}\right)^3\left(-\frac{1}{z^2}\right)^3 .\Delta(L)^{-8/3}(w^2-z^2)^3\prod_{k=1}^{3} c_k C_k \\ &= \left(\frac{z^2-w^2}{w^2z^2}\right)^4 (\prod_{k=1}^{3} c_k C_k)\Delta(L)^{-8/3} \end{aligned}$$

Thus

$$c(L)^{12} = \frac{N}{D} \sim \frac{\Delta(L)^2}{4^4(\prod_{k=1}^3 c_k C_k)}.$$

But

$$\begin{aligned} \wp'(w+t_i) &= 2c_i w + \text{h.o.t.} \\ \wp'(w) &= \frac{-2}{w^3} + \text{h.o.t.} \end{aligned}$$

We have the classical identity ([1], App., Lemma 7)

$$\wp'(w)\wp'(w+t_1)\wp'(w+t_2)\wp'(w+t_3) = \Delta(L)$$

from which it follows that

$$c_1c_2c_3 = -\frac{\Delta(L)}{16}.$$

Also,

$$C_1C_2C_3 = \prod_{j\neq k}(\wp(t_k)-\wp(t_j)) = -\prod_{j>k}(\wp(t_k)-\wp(t_j))^2.$$

Define the constants g_2 and g_3 as usual, by the equation

$$\begin{aligned} \wp'(z)^2 &= 4(\wp(z)-\wp(t_1))(\wp(z)-\wp(t_2))(\wp(z)-\wp(t_3)) \\ &= 4\wp(z)^3 - g_2\wp(z) - g_3. \end{aligned}$$

Then $\Delta(L)$ is by definition $g_2^3 - 27g_3^2$. One easily calculates

$$C_1C_2C_3 = -\frac{\Delta(L)}{16}.$$

It follows that

$$c(L)^{12} = \frac{N}{D} \sim 1$$

as required.

Now let $\psi : \mathbb{C}/L \twoheadrightarrow \mathbb{C}/L'$ be an isogeny. Let

$$\nu(\psi) = \frac{\gamma(\psi z, \psi w, 0; L')}{\prod_{t \in \ker \psi} \gamma(z, w, t; L)}.$$

Again, $\nu(\psi)$ is constant.

Lemma 1.6 *If* $\mathbb{C}/L \xrightarrow{\psi} \mathbb{C}/L' \xrightarrow{\psi'} \mathbb{C}/L''$, *then*

$$\nu(\psi'\psi) = \nu(\psi)^{\deg \psi'} \nu(\psi').$$

The proof is analogous to that of Lemma 1.1.

Corollary 1.7 *Let* $\mu \in \mathbb{Z}_{>0}$. *Then* $\nu(\psi)^{\deg(\mu)-1} = 1$.

Proof. One sees that

$$[\mu]_{L'} \circ \psi = \psi \circ [\mu]_L.$$

$\nu([\mu]) = 1$ as $[\mu]$ is an endomorphism. Then one applies Lemma 1.6, and the result follows.

To complete the proof, one chooses again $\mu = 2, 3$, to see that $\nu(\psi) = 1$. This gives the distribution relation for isogenies also.

2 The other symbols

Write $\{\{x, y\} \oplus t\}$ for the divisor $(x+y+t)+(x-y+t)-2(x+t)-2(y+t)$ (and, more generally, if X is a divisor which corresponds to a symbol, we will write $\{X \oplus t\}$ for the same divisor, but adding t to each point).

For the other symbols, take the derivations of the relation of §1.

Proposition 2.1 *The distribution relation is valid for the symbol $\{x, x\}$, where $2x \neq \mathcal{O}$.*

Proof. One again has the identity

$$\prod_{t \in \ker \psi} \varphi(\{\{x, y\} \oplus t\}, L) = \varphi(\{x, y\}, L').$$

For the symbol $\{x, x\}$, where $2x \neq \mathcal{O}$, put $x = y + \delta y$, and one lets $\delta y \to 0$.

One knows now that

$$\prod_{t_1, t_2 \in \ker \psi} \varphi(\{y + \delta y + t_1, y + t_2\}, L) = [\varphi\{y + \delta y, y\}, L')]^{\deg \psi}.$$

But

$$\begin{aligned}
&\prod_{t_1, t_2 \in \ker \psi} \left[\frac{\varphi(2y + \delta y + t_1 + t_2, L)\varphi(t_1 - t_2 + \delta y, L)}{\varphi(y + \delta y + t_1, L)^2 \varphi(y + t_2, L)^2}\right] \\
= &\left[\prod_{t \in \ker \psi} \frac{\varphi(2y + \delta y + t, L)\varphi(t + \delta y, L)}{\varphi(y + \delta y + t, L)^2 \varphi(y + t, L)^2}\right]^{\deg \psi} \\
= &\left[\varphi(\delta y, L). \prod_{t \in \ker \psi} \frac{\varphi(2y + \delta y + t, L)}{\varphi(y + \delta y + t, L)^2 \varphi(y + t, L)^2}. \prod_{t \in \ker \psi \setminus \{\mathcal{O}\}} \varphi(t + \delta y, L)\right]^{\deg \psi}
\end{aligned}$$

One also has

$$\begin{aligned} & [\varphi(\{y+\delta y, y\}, L')]^{\deg\psi} \\ = & \left[\frac{\varphi(2y+\delta y, L')\varphi(\delta y, L')}{\varphi(y+\delta y, L')^2\varphi(y, L')^2}\right]^{\deg\psi} \end{aligned}$$

Now one uses the following result of Robert ([4], ch.2, 4.1), to get the result, on taking the limit as $\delta y \to 0$. The result of Robert says that if $L' \supset L$, and if $NL' \subset L$, one has the equality

$$\prod_{t\in\ker\psi\setminus\{\mathcal{O}\}} \varphi(t, L)^{12N} = \left(\frac{\Delta(L')}{\Delta(L)}\right)^N.$$

But, because of the formula

$$\varphi(z, L) = e^{-\frac{1}{2}z\eta(z,L)}\sigma(z, L)\Delta(L)^{1/12},$$

one sees that, if $\delta z \sim 0$,

$$\varphi(\delta z, L) \sim 1.\delta z.\Delta(L)^{1/12},$$

as $\sigma'(0) = 1$. The result follows.

The method for the symbol $\{x, -x\}$ is the same:

Proposition 2.2 *The distribution relation holds for the symbol $\{x, -x\}$, where $2x \neq \mathcal{O}$.*

Proof. One knows that

$$\prod_{t_1,t_2\in\ker\psi} \varphi(\{y+\delta y+t_1, -y+t_2\}, L) = [\varphi\{y+\delta y, -y\}, L')]^{\deg\psi}.$$

But

$$\prod_{t_1,t_2\in\ker\psi}\left[\frac{\varphi(\delta y+t_1+t_2,L)\varphi(2y+t_1-t_2+\delta y,L)}{\varphi(y+\delta y+t_1,L)^2\varphi(-y+t_2,L)^2}\right]$$

$$=\left[\prod_{t\in\ker\psi}\frac{\varphi(\delta y+t,L)\varphi(2y+t+\delta y,L)}{\varphi(y+\delta y+t,L)^2\varphi(-y+t,L)^2}\right]^{\deg\psi}$$

$$=\left[\varphi(\delta y,L).\prod_{t\in\ker\psi}\frac{\varphi(2y+\delta y+t,L)}{\varphi(y+\delta y+t,L)^2\varphi(t-y,L)^2}.\prod_{t\in\ker\psi\setminus\{\mathcal{O}\}}\varphi(t+\delta y,L)\right]^{\deg\psi}$$

One also has

$$[\varphi(\{y+\delta y,-y\},L')]^{\deg\psi}$$

$$=\left[\frac{\varphi(\delta y,L')\varphi(2y+\delta y,L')}{\varphi(y+\delta y,L')^2\varphi(-y,L')^2}\right]^{\deg\psi}$$

One uses the result of Robert to finish the proof.

Finally, there remains the case of the symbol $\{x,x\}$ if $2x=\mathcal{O}$.

Proposition 2.3 *The distribution relation holds for the symbol* $\{x,x\}$*, where* $2x=\mathcal{O}$.

Proof. We have just shown the relation

$$\prod\varphi(\{y,y\},L')=\prod_{t\in\ker\psi}\varphi(\{\{y,y\}\oplus t\},L).$$

The proof works in the same way as for the symbol $\{x,x\}$ when $2x\neq\mathcal{O}$.

But if $2x=\mathcal{O}$, one has $\varphi(2x+\delta x+t,L)=\varphi(\delta x+t,L)$, and:

$$\prod_{t_1,t_2\in\ker\psi}\left[\frac{\varphi(\delta x+t_1+t_2,L)\varphi(t_1-t_2+\delta x,L)}{\varphi(x+\delta x+t_1,L)^2\varphi(x+t_2,L)^2}\right]$$

$$=\left[\prod_{t\in\ker\psi}\frac{\varphi(\delta x+t,L)\varphi(t+\delta x,L)}{\varphi(x+\delta x+t,L)^2\varphi(x+t,L)^2}\right]^{\deg\psi}$$

One also has

$$\begin{aligned} & [\varphi(\{x+\delta x, x\}, L')]^{\deg\psi} \\ = \quad & \left[\frac{\varphi(\delta x, L')^2}{\varphi(x+\delta x, L')^2\varphi(x, L')^2}\right]^{\deg\psi} \end{aligned}$$

One uses the result of Robert to finish the proof.

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